LECTURE: 5-5 THE SUBSTITUTION RULE (PART 3)

Example 1: Doing (some) substitutions quickly. In later Calculus courses (Calculus 2 especially) it is quite useful to be able to do some very simple substitutions without having to go through writing out u and du. Do the following integrals using substitution and then see if you can see the pattern well enough to not need to do all of the work.

(a) $\int e^{5x}dx = \frac{1}{5}\int e^u du$

$u = 5x$
$du = 5\,dx$
$\frac{1}{5}du = dx$

$= \frac{1}{5}e^u + C$

$= \boxed{\frac{1}{5}e^{5x} + C}$

Derivative of $y = e^{5x}$ is $y' = 5e^{5x}$
Antiderivative of $y = e^{5x}$ is $1/5\, e^{5x}$

(b) $\int \sin\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi}\int \sin u\, du$

$u = \frac{\pi}{2}x$
$du = \frac{\pi}{2}dx$
$\frac{2}{\pi}du = dx$

$= \frac{2}{\pi}(-\cos u) + C$

$= \boxed{-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + C}$

(c) $\int \sqrt{1-2x}dx = \int\left(-\frac{1}{2}u^{1/2}\right)du$

$u = 1-2x$
$du = -2dx$
$-\frac{1}{2}du = dx$

$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$

$= \boxed{-\frac{1}{3}(1-2x)^{3/2} + C}$

Example 2: Integrate the following functions. Check your answers using a derivative.

a) $\int \sec^2\left(\frac{\pi}{4}\theta\right) d\theta$

$= \boxed{\frac{4}{\pi}\tan\left(\frac{\pi}{4}\theta\right) + C}$

Check:
$\frac{d}{d\theta}\frac{4}{\pi}\tan\left(\frac{\pi}{4}\theta\right) = \frac{4}{\pi}\sec^2\left(\frac{\pi}{4}\theta\right)\cdot\frac{\pi}{4}$
$= \sec^2(\pi/4\,\theta)$ ✓

b) $\int \sec(2x)\tan(2x)dx$

$= \boxed{\frac{1}{2}\sec(2x) + C}$

Check:
$\frac{d}{dx}\frac{1}{2}\sec(2x) = \frac{1}{2}\sec(2x)\tan(2x)\cdot 2$
$= \sec(2x)\tan(2x)$ ✓

c) $\int \sqrt{1+4x}dx = \int (1+4x)^{1/2}dx$

$= \frac{1}{4}\cdot\frac{2}{3}(1+4x)^{3/2} + C$

$= \boxed{\frac{1}{6}(1+4x)^{3/2} + C}$

Check:
$\frac{d}{dx}\frac{1}{6}(1+4x)^{3/2} = \frac{1}{6}\cdot\frac{3}{2}(1+4x)^{1/2}\cdot 4$
$= \frac{12}{12}\sqrt{1+4x}$
$= \sqrt{1+4x}$ ✓

Example 3: Evaluate the following indefinite integrals.

(a) $\int \tan^2 x \sec^2 x dx = \int u^2 du$

$u = \tan x$
$du = \sec^2 x\, dx$

$= \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{3}\tan^3 x + C}$

(b) $\int t^2\cos(1-t^3)dt$

$u = 1-t^3$
$du = -3t^2 dt$
$-\frac{1}{3}du = t^2 dt$

$= \int\left(-\frac{1}{3}\cos(u)\right)du$

$= -\frac{1}{3}\sin(u) + C$

$= \boxed{-\frac{1}{3}\sin(1-t^3) + C}$

Example 4: Evaluate $\int x^3(1-x^2)^{3/2}dx$ $= \int x^2 (1-x^2)^{3/2} \cdot x\, dx$

$u = 1-x^2$
$du = -2x\, dx$
$-\frac{1}{2} du = x\, dx$
$x^2 + u = 1$
$x^2 = u - 1$

$$= \int (u-1) u^{3/2} (-1/2)\, du$$

$$= -\frac{1}{2} \int (u^{5/2} - u^{3/2})\, du$$

$$= -\frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{2}{5}u^{5/2}\right) + C$$

$$= \boxed{-\frac{1}{7}(1-x^2)^{7/2} + \frac{1}{5}(1-x^2)^{5/2} + C}$$

Example 5: Evaluate the following definite integrals.

(a) $\int_0^1 \cos(\pi t)dt$ $= \frac{1}{\pi} \sin(\pi t) \Big|_0^1$

$$= \frac{1}{\pi}(\sin\pi - \sin 0)$$

$$= \boxed{0}$$

(b) $\int_0^{\pi/4} \sin(4x)dx$ $= -\frac{1}{4}\cos(4x) \Big|_0^{\pi/4}$

$$= -\frac{1}{4}\cos\pi + \frac{1}{4}\cos 0$$

$$= 1/4 + 1/4$$

$$= \boxed{1/2}$$

Example 6: Evaluate $\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}dx$. In doing so, change the bounds.

$u = 1 + 1/x$
$du = -1/x^2\, dx$
$-du = 1/x^2\, dx$
$x = 1,\ u = 2$
$x = 4,\ u = 1 + 1/4 = 5/4$

$$= -\int_2^{5/4} u^{1/2}\, du$$

$$= -\frac{2}{3} u^{3/2} \Big|_2^{5/4}$$

$$= -2/3\left(\left(5/4\right)^{3/2} - 2^{3/2}\right)$$

$$= -\frac{2}{3}\left(\frac{\sqrt{125}}{8} - \sqrt{8}\right)$$

$$= \boxed{-\frac{2}{3}\left(\frac{5\sqrt{5}}{8} - 2\sqrt{2}\right)}$$

Example 7: Evaluate the following integrals.

(a) $\int \frac{x}{x^2+4}dx = \int \frac{(1/2)}{u}\,du$

Sub
$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$= \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|x^2+4| + c$

$= \boxed{\frac{1}{2}\ln(x^2+4) + c}$

(b) $\int \frac{x}{\sqrt{25-x^2}}dx = \int \frac{(-1/2)}{\sqrt{u}}\,du$

Sub:
$u = 25-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int u^{-1/2}\,du$

$= -\frac{1}{2}\,\frac{2}{1}u^{1/2} + c$

$= \boxed{-\sqrt{25-x^2} + c}$

Example 8: Evaluate the following integrals.

(a) $\int xe^{-x^2}dx = \int\left(-\frac{1}{2}e^u\,du\right)$

Sub:
$u = -x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}e^u + c$

$= \boxed{-\frac{1}{2}e^{-x^2} + c}$

(b) $\int_1^e \frac{(\ln x)^3}{x}dx = \int_0^1 u^3\,du$

Sub:
$u = \ln x$
$du = 1/x\,dx$
$x=1,\ u=\ln 1 = 0$
$x=e,\ u=\ln e = 1$

$= \frac{1}{4}u^4\Big|_0^1$

$= \boxed{\frac{1}{4}}$

Example 9: Evaluate $\int_{-3}^{3}(x+5)\sqrt{9-x^2}$ dx

$$= \int_{-3}^{3} x\sqrt{9-x^2}\,dx + 5\int_{-3}^{3}\sqrt{9-x^2}\,dx$$

$$= \int_0^0 \left(-\tfrac{1}{2}u^{\frac{1}{2}}\right)du + 5\cdot\tfrac{1}{2}\pi(3)^2$$

$$= 0 + \tfrac{5}{2}\pi\cdot 9$$

$$= \boxed{\tfrac{45\pi}{2}}$$

Int #1

$u = 9-x^2$

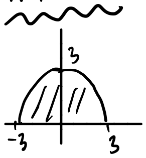
$du = -2x\,dx$

$-\tfrac{1}{2}du = x\,dx$

$x=3,\ u=0$

$x=-3,\ u=0$

Int #2



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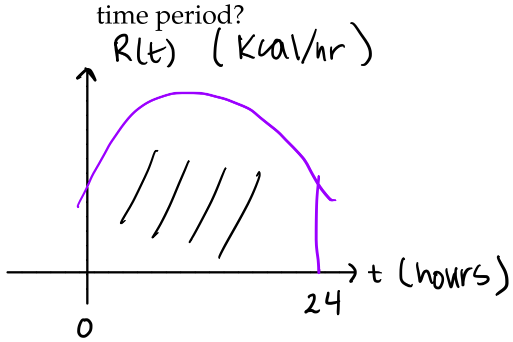
Example 10: A model for the basal metabolic rate, in kcal/h, of a young man is $R(t) = 85 - 0.2\cos(\pi t/12)$, where t is the time in hours measured from 5:00 AM. What is the total basal metabolic rate of this man over a 24 hour time period?



Area has units kcal.

$$A = \int_0^{24}\left(85 - 0.2\cos(\pi t/12)\right)dt$$

$$= \left(85t - \tfrac{2}{10}\cdot\tfrac{12}{\pi}\sin(\pi t/12)\right)\Big|_0^{24}$$

$$= 85(24) - \tfrac{12}{5\pi}\sin(2\pi) - (0-0)$$

$$= \boxed{2040 \text{ kcal}}$$